ON THE EQUILIBRIUM AND PROPAGATION OF CRACKS IN AN ANISOTROPIC MEDIUM

(O RAVNOVESII I RASPROSTRANENII TRESHCHIN V ANIZOTROPNOI SREDE)

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On the basis of the concepts presented in [1,2], some problems of equilibrium and propagation of rectilinear cracks in an anisotropic medium are studied.*

1. Fundamental relations. Let us study the motion of an elastic anisotropic medium under the conditions of plane strain. The equations of motion are

$$\frac{\partial \sigma_{i\alpha}}{\partial x_\alpha} = \rho \frac{\partial^2 u_i}{\partial t^2} \qquad (i = 1, 2) \tag{1.1}$$

Here and throughout this paper summation is implied by repeated Greek indices having the values of one and two; $\sigma_{i\alpha}$ are the components of the stress tensor, u_i are the components of the displacement vector, x_α are rectangular Cartesian coordinates, t is the time, and ρ is the density of the medium. For an anisotropic body, where the plane $x_1 x_2$ is the plane of elastic symmetry, the generalized Hooke's law has the form [3]

$$\sigma_{ij} = b_{ij\beta\gamma}\varepsilon_{\beta\gamma} \qquad \left(\varepsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]\right) \tag{1.2}$$

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* The authors would like to take this opportunity to introduce some clarification in [2]. Formula (4.4) should be written in the following form ($h = \sqrt{(R^2 - x^2)}$)

$$\delta_2 W = \iint\limits_{\delta S} \sigma_y v\, dS = \frac{(1-\nu^2)N^2}{\pi^2 E}\int\limits_{-R}^{R} dx \int\limits_0^h \sqrt{\frac{R-y}{y}}\, dy = \frac{(1-\nu^2)N^2\delta S}{2\pi E}$$

The above correction is of no significance to the subsequent presentation.

Here ϵ_{ij} are components of the strain tensor. The quantities $b_{ij\beta\gamma}$ represent the elastic constants of the material, where

$$b_{ij\beta\gamma} = b_{\gamma\beta ji} = b_{ij\gamma\beta} = b_{ji\gamma\beta}$$

Thus, in the general case the studied body is characterized by six independent constants, which we shall choose to be b_{1111}, b_{1112}, b_{1122}, b_{1212}, b_{2122}, b_{2222}. In the orthotropic body $b_{1112} = b_{2122} = 0$.

For the isotropic body the following is also true: $b_{1111} = b_{2222}$, $b_{1122} = b_{1111} - 2b_{1212}$.

After substitution of (1.2) into (1.1) we obtain the fundamental dynamic equation

$$L_{i\alpha}u_\alpha = 0, \quad L_{ij} = \frac{1}{2}(b_{i\alpha\beta i} + b_{i\alpha j\beta}) \frac{\partial^2}{\partial x_\alpha \partial x_\beta} - \rho \frac{\partial^2}{\partial t^2}\delta_{ij} \tag{1.3}$$

where δ_{ij} is the Kronecker delta, so that $L_{ij} = L_{ji}$.

The general solution of the system of equations (1.3) has the form

$$u_1 = L_{22}\Psi_2 - L_{12}\Psi_1, \qquad u_2 = L_{11}\Psi_1 - L_{12}\Psi_2 \tag{1.4}$$

where the functions Ψ_1 and Ψ_2 satisfy the equation

$$(L_{11}L_{22} - L_{12}^2)\Psi = 0 \tag{1.5}$$

For the purposes we have in mind here it is sufficient to study the case $\Psi_1 = \Psi$, $\Psi_2 = 0$. Below we study in different versions the mixed problem of the dynamic theory of elasticity for the anisotropic half-plane, which is stationary in a system of coordinates ξ_1, ξ_2 moving with a constant velocity v in the direction of the negative x_1-axis:

$$\xi_i = x_i + vt\delta_{i1} \tag{1.6}$$

(as a special case, the static problems are obtained for $v = 0$).

In the stationary case, the function Ψ depends only on ξ_1 and ξ_2, and does not depend explicitly on time, so that

$$L_{ij}\Psi = A_{ij\alpha\beta}\frac{\partial^2\Psi}{\partial\xi_\alpha\partial\xi_\beta}, \qquad A_{ij\alpha\beta} = \tfrac{1}{2}(b_{i\alpha\beta j} + b_{i\alpha j\beta}) - \rho v^2\delta_{ij}\delta_{1\alpha}\delta_{1\beta} \tag{1.7}$$

and the basic equation (1.5) becomes

$$B_{\alpha\beta\gamma\varepsilon}\frac{\partial^4\Psi}{\partial\xi_\alpha\partial\xi_\beta\partial\xi_\gamma\partial\xi_\varepsilon} = 0, \qquad B_{\alpha\beta\gamma\varepsilon} = A_{11\alpha\beta}A_{22\gamma\varepsilon} - A_{12\alpha\beta}A_{12\gamma\varepsilon} \tag{1.8}$$

The corresponding characteristic equation can be written in the form

$$B_{\alpha\beta\gamma\varepsilon}\mu^{\delta_{\alpha1}+\delta_{\beta1}+\delta_{\gamma1}+\delta_{\varepsilon1}} = 0 \tag{1.9}$$

The subsequent analysis will be only restricted to the elliptic case, where there are no real roots of (1.9). As was shown by Lekhnitskii [4], the static problem always corresponds to the elliptic case. Because of continuity considerations, the elliptic quality also holds for sufficiently small velocities. For an orthotropic body, if the boundary of the half-plane is a line of elastic symmetry, Equation (1.9) becomes biquadratic:

$$L\mu^4 + M\mu^2 + N = 0 \tag{1.10}$$

where

$$L = b_{1212}b_{2222}, \qquad N = (b_{1111} - \rho v^2)(b_{1212} - \rho v^2)$$

$$M = b_{1111}b_{2222} - b_{1122}^2 - 2b_{1212}b_{1122} - \rho v^2 (b_{1212} + b_{2222})$$

Note that in the case of the orthotropic body the roots of the characteristic equation are not necessarily purely imaginary.

Using the method for the static problem of the two-dimensional theory of elasticity of an anisotropic medium proposed by Lekhnitskii [5], and later applied by Galin [6] to the problem of the punch which moves along the boundary of an isotropic half-plane, we shall write the general solution of Equation (1.8) in the form

$$\Psi = 2\,\mathrm{Re}\,[F_1(z_1) + F_2(z_2)], \qquad z_i = \xi_1 + \mu_i\xi_2 \tag{1.11}$$

where F_1, F_2 are arbitrary analytic functions, and μ_1, μ_2, $\bar{\mu}_1$, $\bar{\mu}_2$ are the roots of the characteristic equation. By substituting (1.11) into (1.4) and (1.2) we obtain expressions for displacements and stresses of the form

$$u_i = 2\,\mathrm{Re}\,[d_{i\alpha}\varphi_\alpha(z_\alpha)],\ \sigma_{ij} = 2\,\mathrm{Re}\,[e_{ij\alpha}\varphi_\alpha'(z_\alpha)] \quad (\varphi_j(z_j) = F_j''(z_j)) \tag{1.12}$$

Here the coefficients d_{ij}, e_{ijk} are given by the formulas

$$d_{1j} = -b_{1112} - (b_{1122} + b_{1212})\mu_j - b_{1222}\mu_j^2 \tag{1.13}$$

$$d_{2j} = b_{1111} - \rho v^2 + 2b_{1112}\mu_j + b_{1212}\mu_j^2$$

$$e_{11j} = \mu_j(b_{1112}^2 - b_{1111}b_{1212}) + \mu_j^2(b_{1112}b_{1122} - b_{1111}b_{1222}) + \\ + \mu_j^3(b_{1122}b_{1212} - b_{1112}b_{1222}) - \rho v^2(b_{1112} + \mu_j b_{1122})$$

$$e_{12j} = (b_{1111}b_{1212} - b_{1112}^2) - \rho v^2 (b_{1212} + \mu_j b_{2122}) + \\ + \mu_j [b_{1111}b_{2122} - b_{1122}b_{1112} + \mu_j (b_{1112}b_{2122} - b_{1212}b_{1122})]$$

$$e_{22j} = (b_{2122}b_{1111} - b_{1112}b_{1122}) + \mu_j [b_{2122}b_{1112} - b_{1122}(b_{1122} + b_{1212}) + \\ + b_{1111}b_{2222} + 2\mu_j b_{2222}b_{1112} + \mu_j^2 (b_{1212}b_{2222} - b_{2122}^2)] - \rho v^2 (b_{2122} + \mu_j b_{2222})$$

2. The general problem for the half-plane. Rayleigh surface waves. The moving punch. 1. Assume that on the boundary of the lower half-plane $\xi_2 \leqslant 0$ normal and shear stresses, distributed in some manner, are applied, and these distributions of stresses move uniformly along the boundary of the half-plane with a velocity v.

According to Galin [6], we introduce the analytic functions

$$w_1(z) = \int_{-\infty}^{\infty} \frac{\sigma(\zeta)\,d\zeta}{\zeta - z} = U_1 - iV_1, \quad w_2(z) = \int_{-\infty}^{\infty} \frac{\tau(\zeta)\,d\zeta}{\zeta - z} = U_2 - iV_2 \quad (2.1)$$

where $\sigma(\xi_1)$ and $\tau(\xi_1)$ are the distributions of the normal and shear stresses at the boundary, respectively. We have

$$\tau(\xi_1) = 2\,\mathrm{Re}\,[e_{121}\varphi_1'(\xi_1) + e_{122}\varphi_2'(\xi_1)]$$
$$\sigma(\xi_1) = 2\,\mathrm{Re}\,[e_{221}\varphi_1'(\xi_1) + e_{222}\varphi_2'(\xi_1)] \quad (2.2)$$

From this and from (2.1), we obtain

$$e_{121}\varphi_1'(z) + e_{122}\varphi_2'(z) = \frac{1}{4\pi i} w_2(z), \quad e_{221}\varphi_1'(z) + e_{222}\varphi_2'(z) = \frac{1}{4\pi i} w_1(z)$$

When we solve this system with respect to $\phi_1'(z)$ and $\phi_2'(z)$ we find

$$\varphi_1'(z) = \frac{1}{4\pi i\Delta} [e_{222}w_2(z) - e_{122}w_1(z)], \quad \varphi_2'(z) = -\frac{1}{4\pi i\Delta} [e_{221}w_2(z) - e_{121}w_1(z)] \quad (2.3)$$

where

$$\Delta = e_{121}e_{222} - e_{122}e_{221} \quad (2.4)$$

Differentation of (1.12) with respect to ξ_1 and going to the limit as $\xi_2 = -0$ results in

$$\left(\frac{\partial u_1}{\partial \xi_1}\right)_{\xi_2=0} = \mathrm{Re}\left[\frac{d_{12}e_{121} - d_{11}e_{122}}{2\pi i\Delta} w_1(\xi_1) + \frac{d_{11}e_{222} - d_{12}e_{221}}{2\pi i\Delta} w_2(\xi_1)\right] \quad (2.5)$$

$$\left(\frac{\partial u_2}{\partial \xi_1}\right)_{\xi_2=0} = \mathrm{Re}\left[\frac{d_{22}e_{121} - d_{21}e_{122}}{2\pi i\Delta} w_1(\xi_1) + \frac{d_{21}e_{222} - d_{22}e_{221}}{2\pi i\Delta} w_2(\xi_1)\right] \quad (2.6)$$

where $w_1(\xi_1)$ and $w_2(\xi_2)$ are the limiting values of the functions when the points on the abscissa are approached from below, and, according to the formulas of Sokhotskii-Plemelj [7], are equal to

$$u_1(\xi_1) = \text{v. p.} \int_{-\infty}^{\infty} \frac{\sigma(\zeta)\, d\zeta}{\zeta - \xi_1} - i\pi\sigma(\xi_1), \; w_2(\xi_1) = \text{v. p.} \int_{-\infty}^{\infty} \frac{\tau(\zeta)\, d\zeta}{\zeta - \xi_1} - i\pi\tau(\xi_1) \quad (2.7)$$

Formulas (2.5), (2.6) and (2.7) allow us to reduce the stationary mixed problem of the dynamic theory of elasticity for the anisotropic half-plane to the well-studied Hilbert problem of the theory of analytic functions (the methods of solution of the Hilbert problem can be found in the monographs of Muskhelishvili [8] and Gakhov [9]).

In the particular case of the orthotropic body, the boundary of the half-plane being a line of elastic symmetry, the quantities

$$C = \frac{d_{22}e_{121} - d_{21}e_{122}}{2\pi i\Delta}, \qquad D = \frac{d_{22}e_{221} - d_{21}e_{222}}{2\pi\Delta} \quad (2.8)$$

are real, even if the roots of the characteristic equation (1.10) are not purely imaginary, so that Formula (2.6) becomes

$$\left(\frac{\partial u_2}{\partial \xi_1}\right)_{\xi_2=0} = CU_1(\xi_1) + DV_2(\xi_1) \quad (2.9)$$

2. Keeping in mind future usefulness, let us study as an example the surface waves at the boundary of an anisotropic half-plane. This problem was studied by a number of authors by means of other methods; a review and discussion of these papers from one point of view is given by Scholte [10].

If an instantaneous disturbance is created on the free surface of a half-space at rest, then a long time after the creation of this disturbance, the dilatational waves go to infinity and damp out there. There remain only the surface waves (if they exist) which progress, without changing their form, along the boundary of the half-space with a constant velocity v.

The study of the surface waves is a simple case of the general mixed problem formulated before. From the condition of the absence of normal and shear stresses at the free surface and from Equations (1.13) we find

$$\text{Re}\,[e_{121}\varphi_1'(\xi_1) + e_{122}\varphi_2'(\xi_1)] = 0, \qquad \text{Re}\,[e_{221}\varphi_1'(\xi_1) + e_{222}\varphi_1(\xi_1)] = 0 \quad (2.10)$$

In order to satisfy the boundary conditions (2.10) by non-trivial solutions it is necessary to satisfy the following condition:

$$\Delta = e_{121}e_{222} - e_{122}e_{221} = 0 \quad (2.11)$$

Together with the characteristic equation (1.9), relation (2.11) determines the velocity of propagation of the surface waves, if these waves exist. In the case of the orthotropic body, if the boundary of the half-space is a plane of elastic symmetry, the characteristic equation (1.10) is solved explicitly. By substituting its solution into the appropriate equation (2.11) we obtain the equation for the velocity of propagation of surface waves in the form

$$PR - PS\frac{M}{L} + (PS + QR)\sqrt{\frac{N}{L}} - QS\frac{N}{L} = 0 \qquad (2.12)$$

Here

$$P = b_{1111} - \rho v^2, \qquad Q = b_{1122}$$
$$R = b_{2222}(b_{1111} - \rho v^2) - b_{2211}(b_{1212} + b_{1122})$$
$$S = b_{1212}b_{2222}$$

and the quantities L, M, N are the coefficients of the characteristic equation (1.10). In the case of the isotropic body we obtain from here the ordinary Rayleigh equation [3]

$$\left(1 - \frac{1}{2}m^2\right)^2 - \sqrt{1 - m^2}\sqrt{1 - \frac{1 - 2\nu}{2(1 - \nu)}m^2} = 0, \qquad m = \frac{v}{c_2} \quad (2.13)$$

where c_2 is the velocity of propagation of the deformation waves in the body and ν is Poisson's ratio. Equation (2.13) is known to have a unique real root $m_0 < 1$ for $-1 < \nu \leqslant 1/2$. In the case of a general type of isotropy Equations (1.9) and (2.11) give complete values of v^2. This means that with an arbitrary anisotropy surface waves do not exist. Of great interest is the complete analysis of the cases, so far not carried out, where Equation (2.12) has a real root, i.e. the cases where surface waves exist at the boundary of the orthotropic body. Note that from the existence proof of a unique positive root of the Rayleigh equation [11] and from the continuity expressions follows directly the existence of a unique positive root of Equation (2.12) for slightly anisotropic bodies.

3. As a second example, let us study the problem of the punch which moves along the boundary $\xi_2 = 0$ of an anisotropic elastic half-plane, taking into account Coulomb friction at the boundary of contact of the punch and the body. The boundary conditions for this problem have, of course, the form

$$\sigma_{12} = \sigma_{22} = 0 \qquad (-\infty < \xi_1 < a,\ b < \xi_1 < \infty) \qquad (2.14)$$
$$\sigma_{12} = k\sigma_{22}, \qquad \frac{\partial u_2}{\partial \xi_1} = f'(\xi_1), \qquad \int_a^b \sigma_{22}(\zeta)\,d\zeta = P \qquad (a \leqslant \xi_1 \leqslant b)$$

where a and b are the coordinates of the boundary points of the line of contact of the punch and the half-plane, k is the coefficient of Coulomb friction, $f(t)$ is a function describing the form of the punch, and P is the force pressing the punch against the body. The corresponding boundary conditions of the Hilbert problem for the determination of the function $w_1(z) = w_2(z)/k$ are found from Formulas (2.6) and (2.7) to be

$$V_1 = 0 \quad (-\infty < \xi_1 < a,\ b < \xi_1 < \infty)$$

$$\operatorname{Re}\{(C + ikD)\, w_1(\xi_1)\} = f'(\xi_1) \quad (a \leqslant \xi_1 \leqslant b) \tag{2.15}$$

The parameters C and D, which are determined from the expressions (2.8) are complex in the case of anisotropy of the general form. We determine the constants p and q from the relations

$$\operatorname{Re}(C + ikD) = \frac{1}{\pi p}, \qquad \operatorname{Im}(C + ikD) = \frac{kq}{\pi p} \tag{2.16}$$

Then the second condition of (2.15) can be rewritten in the form

$$\pi p f'(\xi_1) = U_1 + kqV_1 \tag{2.17}$$

so that for the determination of the function $w_1(z)$ we obtain the same boundary problem as in the case of the punch which moves along an isotropic half-plane [6]. The solution of this boundary-value problem can be found in [6]. By using Formulas (1.12) and expressing the functions $\phi_j'(z)$ in terms of $w_1(z)$ by means of Formulas (2.3), one can write the stresses in the elastic body in the form

$$\sigma_{ij} = \frac{\Phi(\xi_1, \xi_2)}{\Delta}$$

where Φ is some function which remains finite as the speed of the punch approaches the speed of the Rayleigh waves, if such waves exist, whereas the quantity Δ, which is determined from Equation (2.4), tends at the same time to zero. Thus, just as in the case of the isotropic body [12], when the speed of motion of the punch approaches the speed of the surface waves, if such waves exist, unusual resonance phenomena appear, which are connected with the unlimited growth of the stresses in the elastic body. Actually, this is connected with a radical change of the motion at near-Rayleigh velocities, which limits the statement of the present problem in terms of sub-Rayleigh velocities. If the character of the anisotropy is such that surface waves do not exist, then the resonance does not appear and the adopted formulation of the problem as well as the method of solution are applicable up to the maximum velocities, which requires an elliptic form for Equation (1.8).

3. The isolated rectilinear crack in an orthotropic body. Let us study the isolated rectilinear crack in an orthotropic infinite body under the conditions of plane strain, which propagates along some line of elastic symmetry which we shall call x_1. The crack is maintained in the open state by some system of loads which is symmetric with respect to the x_1-axis. Completely analogously to the corresponding analysis for the isotropic body [1] it is sufficient to study the case where the crack is maintained by symmetric normal stresses $-p(x_1)$, which are equal in magnitude and opposite in direction to the tearing stresses $p(x_1)$, which would exist in place of the crack in a continuous body. It is natural to assume the crack to be fixed regardless of the fact that the relations given below yield a solution for the problem of a crack moving with a constant velocity v and only as a special case, for a fixed crack. The problem of a moving crack of fixed length in a uniform field was studied for the isotropic body by Yoffe [13]; however, even the statement of such a problem appears to be physically unrealistic.

The problem studied here is symmetric with respect to the crack line, and thus it is sufficient to analyse only the lower half-plane $\xi_2 \leqslant 0$. The corresponding boundary-value problem of the theory of elasticity for the lower half-plane $\xi_2 \leqslant 0$ is formulated in the following fashion:

$$\begin{gathered} u_2 = 0, \qquad \sigma_{12} = 0 \qquad (-\infty < \xi_1 < a,\ b < \xi_1 < \infty) \\ \sigma_{22} = -g(\xi_1), \qquad \sigma_{12} = 0 \qquad (a < \xi_1 < b) \end{gathered} \tag{3.1}$$

Here $-g(\xi_1)$ is the distribution of the acting loads and forces of cohesion. In the case at hand the function $w_2(z)$ is identically equal to zero. From the relations (2.6), (2.7), and (3.1) we obtain the boundary conditions of the Hilbert problem for the determination of the function $w_1(z)$:

$$\begin{gathered} \operatorname{Re}[Cw_1(\xi_1)] = 0 \qquad (-\infty < \xi_1 \leqslant a,\ b \leqslant \xi_1 < \infty) \\ V_1 = -\pi g(\xi_1) \qquad (a < \xi_1 < b) \end{gathered} \tag{3.2}$$

For the orthotropic body and a crack that propagates along a line of elastic symmetry, the constant C, given by Formula (2.8), is real, and can thus be cancelled. For the determination of the function $w_1(z)$ the same boundary problem is obtained as in the case of the isotropic body. The difference appears only later in the expressions for the stresses and displacements. According to the Keldysh-Sedov formula [7], we have

$$w_1(z) = -\frac{1}{\sqrt{(z-a)(z-b)}} \int_a^b \frac{\sqrt{(t-a)(t-b)}\, g(t)\, dt}{t-z} \tag{3.3}$$

The fundamental hypotheses on the smallness and the autonomy of the end region of the crack, where forces of molecular cohesion are acting [1,2], and the condition of finite stresses at the ends of the crack can be applied in the present case just as in the case of the isotropic body. Thus, similarly to [1], the condition which determined the position of the ends of the cracks is valid, i.e. the tearing stress σ_{22} near the end of the crack, computed disregarding the forces of molecular cohesion, approaches infinity as

$$\sigma_{22} = \frac{K}{\pi\sqrt{s}} \qquad \left(K = \int_0^d \frac{G(t)dt}{\sqrt{t}}\right) \tag{3.4}$$

Here s is the distance from the end of the crack, K is the cohesion modulus [1], $G(t)$ is the distribution of the forces of molecular cohesion in the end region of the crack, where these forces are acting, and d is the longitudinal dimension of the end region.

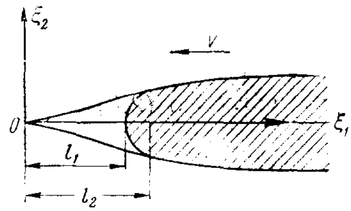
Condition (3.4) holds for all equilibrium cracks in the orthotropic bodies, which lie along the line of elastic symmetry. Note that in contrast to the case of the isotropic body, the value of the constant k depends on which of the planes of elastic symmetry the crack lies on.

In particular, in the present case of the isolated equilibrium crack, the conditions which determine the ends of the cracks a and b have the form

$$\int_a^b p(t)\sqrt{\frac{t-a}{b-t}}\,dt = K\sqrt{b-a}, \qquad \int_a^b p(t)\sqrt{\frac{b-t}{t-a}}\,dt = K\sqrt{b-a} \tag{3.5}$$

Superficially, these conditions coincide with the corresponding conditions for the isotropic body [1]. The difference appears in the fact that with an application of tearing stresses inside the body instead of at the surface of the crack, the distribution $p(x_1)$ for the anisotropic body differs considerably from the distribution for the isotropic body. Further, the cohesion modulus K depends on the direction of the crack. Note that the problem of the isolated rectilinear crack in an anisotropic body was studied by Stroh [14]. However, because of his complicated energy approach, Stroh did not obtain a final solution.

4. The cleavage of an anisotropic body. 1. Assume that an orthotropic body with planes of elastic symmetry and parallel axes x_1 and x_2 is wedged open under the conditions of plane strain by a thin, absolutely rigid, infinite wedge which moves with a constant velocity v in the direction of the negative x_1-axis. In front of the wedge a free crack is formed. We choose as the origin of the coordinate system the point of closure of the crack (see figure). Coulomb friction forces are acting at the surface of contact of the wedge and the splitting body.



Because of the symmetry of the problem with respect to the ξ_1-axis one can study the motion in only the lower half-plane $\xi_2 \leqslant 0$. The fact that the wedge is thin allows us to bring the boundary conditions down to the ξ_1-axis. Thus, the boundary conditions of the corresponding mixed problem of the dynamic theory of elasticity for the lower half-plane can be represented in the following manner:

$$u_2 = 0, \qquad \sigma_{12} = 0 \qquad (-\infty < \xi_1 \leqslant 0)$$
$$\sigma_{12} = \sigma_{22} = 0 \qquad (0 < \xi_1 < l_2) \tag{4.1}$$
$$\sigma_{12} = k\sigma_{22}, \qquad u_2 = -f(\xi_1 - l_1) \qquad (l_2 \leqslant \xi_1 < \infty)$$

Here k is the Coulomb friction coefficient, $f(t)$ is a function describing the form of the wedge in a system of coordinates with its origin at the forward point of the wedge, l_1 is the distance from the forward point of the wedge to the end of the crack, and l_2 is the distance from the initial point of contact of the crack with the wedge to the end of the crack. Using Formulas (2.7) and (2.9), we obtain for the determination of the function $w_1(z)$ the following boundary-value problem:

$$U_1 = 0 \qquad (-\infty < \xi_1 \leqslant 0), \qquad V_1 = 0 \qquad (0 < \xi_1 < l_2)$$
$$CU_1 + DkV_1 = -f'(\xi_1 - l_1) \qquad (l_2 \leqslant \xi_1 < \infty) \tag{4.2}$$

If function $w_1(z)$ is known, then the determination of function $w_2(z)$ in the given case is elementary. Let us recall that for the present conditions the constants C and D, which are given by Equations (2.8), are real. If one introduces the notation

$$p = \frac{1}{\pi C}, \qquad q = \frac{D}{C} \tag{4.3}$$

then the boundary-value problem (4.2) coincides with the corresponding boundary-value problem which was solved earlier in the study of the cleavage of an isotropic body [12]. An analysis of additional conditions, which determine the constants entering into the solution of the elasticity problem, shows that in the solution of the problem of cleavage of an orthotropic body one can utilize the formulas from [12] by choosing in these formulas values of p and q which were determined by relation (4.3), and keeping in mind that the modulus of cohesion of the material depends on the direction in which the cleavage proceeds.

2. Let us look in greater detail at the important problem of the splitting of an orthotropic body by means of an immobile wedge of constant thickness $2h$, neglecting the forces of friction at the sides of the wedge.

Using the results of [12], we obtain the following expression for the length $l = l_1 = l_2$ of a free crack before the wedge:

$$l = \frac{p^2h^2}{K^2} = \frac{h^2}{\pi^2 C_0^2 K^2} \tag{4.4}$$

where C_0 is the value of the constant C given by Formula (2.8) at $v = 0$. We have, according to (2.8) and (1.14)

$$\pi C_0 = \frac{\varepsilon_1 + \varepsilon_2}{2} \frac{\sqrt{b_{1111} b_{2222}}}{b_{1111} b_{2222} - b_{1122}^2} \tag{4.5}$$

where ϵ_1 and ϵ_2 are the roots of the characteristic equation (1.10) divided by i. Their values depend only on the elastic constants of the material

$$\varepsilon_1 = \sqrt{\frac{M_0 - \sqrt{M_0^2 - 4L_0N_0}}{2L_0}} \qquad \varepsilon_2 = \sqrt{\frac{M_0 + \sqrt{M_0^2 - 4L_0N_0}}{2L_0}}$$

Relation (4.4) can be utilized for an experimental determination of the cohesion modulus, as was done in [1] for the isotropic body. A thin wedge of constant thickness, made of a material considerably more rigid than that of the one under study, is driven into a small plate made of the material in question, which is sufficiently thick for the state of stress in it to be assumed to be that of plane strain. The wedge should be driven in until the distance from the end of the wedge to the end of the crack l remains constant, which will indicate that the influence of the ends of the plate is insignificant. By measuring l and knowing the elastic constants of the material we can find the cohesion modulus by means of the formula

$$K = \frac{2h(b_{1111} b_{2222} - b_{1122}^2)}{(\varepsilon_1 + \varepsilon_2)\sqrt{b_{1111} b_{2222}}\sqrt{l}} \tag{4.6}$$

In the case of an isotropic body we obtain the previously known result

$$K = \frac{Eh}{2(1-\nu^2)\sqrt{l}} \tag{4.7}$$

Note that the realization of this experiment in anisotropic materials is simpler than in isotropic ones, since the cracks bend more readily in the latter.

3. Let us return now to the dynamic problem. As was shown in [12], the length of the free part of the crack l_2 tends to zero for $p \to 0$. But from Formulas (4.3) and (2.8) it follows that p is proportional to the determinant Δ and tends to zero when the velocity v of the wedge approaches the speed of the Rayleigh surface waves corresponding to the given direction, if such waves exist. Thus, the length of the free part of the crack tends to zero as the velocity of motion of the wedge approaches the Rayleigh velocity; and thus, just as in the isotropic case, the velocity of propagation of a crack cannot exceed the Rayleigh velocity.

It can be shown completely analogously to [12] that when approaching the Rayleigh velocity the stresses near the end of the crack increase, at which time the tearing stress σ_{11} grows faster then the tearing stress σ_{22}. This shows that when the speed of motion of the wedge approaches the Rayleigh velocity transverse cracks appear and the picture of motion changes considerably. Thus, the present statement of the problem is known to be applicable only for velocities of wedge motion below the Rayleigh velocity.

The upper velocity limit to which the formulation of the cleavage problem adopted in this paper applies, also depends on the ratio of the cohesion moduli in the direction of splitting and in the direction perpendicular to that. For the crack to be rectilinear it is necessary that this ratio be not greater than unity. Otherwise the crack in front of the wedge will curve under the influence of incidental factors even with a motionless wedge. In the frequently encountered case when the cohesion modulus in the direction of the cleavage is considerably smaller than the cohesion modulus in the transverse direction (as, for instance, in the splitting of wood along the fiber) one can assume the rectilinearity of the crack to be assured and the accepted formulation of the problem to be correct up to the wedge velocity equal to the Rayleigh velocity. If the cohesion moduli in the direction of the cleavage and in the transverse direction are equal to each other, then one can show completely analogously to the isotropic case that there exists still another sub-Rayleigh critical velocity, up to which the direction of the cleavage lies along the line of maximum tearing stresses. When this velocity is exceeded, then the crack will begin to curve.

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